VIRILE AGITUR

Number: ____________________

Teacher: ____________________

2013

Trial Higher School Certificate Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen only
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-14, show relevant mathematical reasoning and/or calculations

Teachers:
Mr Bradford
Mr Johansen*
Mr Mulray
Mr Vuletich

Total marks – 70

Section I: Pages 1-3

10 marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II: Pages 4-8

60 marks
- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section

Write your Board of Studies Student Number and your teacher's name on the front cover of each writing booklet

This paper MUST NOT be removed from the examination room

Number of Students in Course: 74

Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Which expression is a correct factorisation of $x^3 - 27$?

(A) $(x-3)(x^2-3x+9)$

(B) $(x-3)(x^2-6x+9)$

(C) $(x-3)(x^2+3x+9)$

(D) $(x-3)(x^2+6x+9)$

2 From six girls and four boys, a committee of 3 girls and 2 boys is to be chosen. How many different committees can be formed?

(A) 26
(B) 120
(C) 252
(D) 1440

3 The term independent of x in the expansion of $\left(x+\frac{2}{x}\right)^6$ is:

(A) 160

(B) 80

(C) 40

(D) 20

4 Consider the function $f(x)=\frac{2x}{x+1}$ and its inverse function $f^{-1}(x)$.

Evaluate $f^{-1}(3)$.

(A) -3

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) 3

5 The solution to $2\sin^2\theta - \sin\theta = 0$ for $0 \le \theta \le \pi$ is:

(A) $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi$

(B) $\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$

(C) $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

(D) $\theta = \frac{\pi}{6} or \frac{5\pi}{6}$

6 An object is projected with a velocity of 30 ms^{-1} at an angle of $\tan^{-1}\left(\frac{3}{4}\right)$ to the horizontal.

What is the initial vertical component of its velocity?

(A) 18 ms^{-1}

(B) 50 ms^{-1}

(C) $30\tan\left(\frac{3}{4}\right)$ ms^{-1}

(D) $30\sin\left(\frac{3}{4}\right)$ ms^{-1}

7 The integral of $\cos^2 2x$ is ?

(A) $\frac{\cos^3 2x}{3} + C$

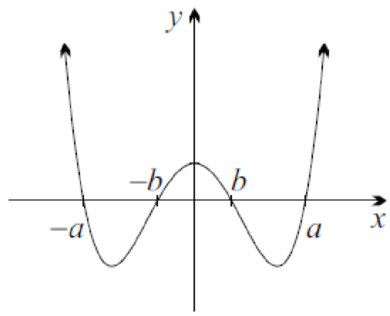
(B) $\left(\frac{\cos 2x}{6}\right)^3 + C$

(C) $\frac{x}{2} + \frac{\sin 4x}{8} + C$

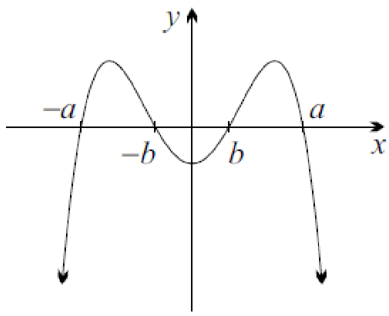
(D) $\frac{x}{2} + \frac{\sin 2x}{4} + C$

8 Which diagram best represents $P(x) = (x-a)^2(b^2 - x^2)$, where $a > b$?

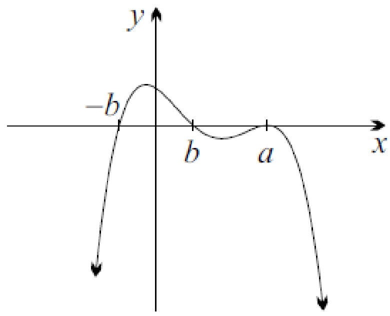
(A)



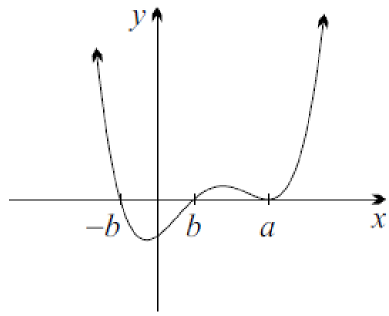
(B)



(C)



(D)



9 Which of the following is the general solution of $2\sin^2\left(6t + \frac{\pi}{4}\right) = 1$?

(A) $t = \frac{n\pi}{3} - \frac{\pi}{6}$ and $t = \frac{n\pi}{3} + \frac{\pi}{12}$, where n is an integer.

(B) $t = \frac{n\pi}{12} - \frac{\pi}{24}$, where n is an integer.

(C) $t = \frac{n\pi}{3}$ and $t = \frac{n\pi}{3} + \frac{\pi}{12}$, where n is an integer.

(D) $t = \frac{n\pi}{12}$, where n is an integer.

10 The equation of the normal to the parabola $x^2 = 4ay$ at the variable point $P(2ap, ap^2)$ is given by $x + py = 2ap + ap^3$.

How many different values of p are there such that the normal passes through the focus of the parabola?

(A) 0

(B) 1

(C) 2

(D) 3

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve the equation $\dfrac{1}{x-3} < 3$. **3**

(b) Evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$, giving your answer in exact form. **2**

(c) Differentiate with respect to x:

(i) $y = \tan^{-1} 2x$ **1**

(ii) $y = \sec^4 x$ **2**

(d) Find, correct to the nearest degree, the acute angle between the lines $y = 3$ and $y = -\dfrac{5}{3}x + 2$. **2**

(e) Let α, β and γ be the roots of $2x^3 - x^2 + 3x - 2 = 0$. Find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **2**

(f) Use the substitution $u = 1 + \tan x$ to evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$. **3**

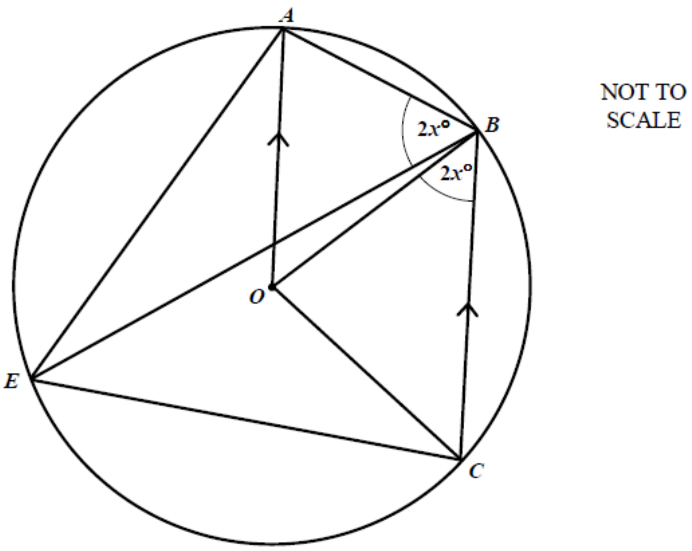
Leave your answer in exact form.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) By considering $f(x) = x - 3\sin x$, show that the curves $y = x$ and $y = 3\sin x$ meet at a point P whose x-coordinate is between $x = 2$ and $x = 3$. **1**

(ii) Use one application of Newton's method, starting at $x = 2$, to find an approximation of the x-coordinate of P. Give your answer correct to two decimal places. **2**

(b) In the diagram, $ABCE$ is a cyclic quadrilateral such that AO is parallel to BC. O is the centre of the circle and $\angle ABE = \angle OBC = 2x^\circ$.



Copy or trace the diagram into your writing booklet.

(i) Prove that $\angle AEB = x^\circ$. **2**

(ii) Prove that $\angle BCE = 3x^\circ$. **2**

Question 12 continues on page 6

(c) Melissa takes a bottle of milk from the refrigerator for baby Henry. To heat the bottle, Melissa puts it in a saucepan of continuously boiling water.

Let y° C be the temperature of the milk time t minutes after the baby's bottle is placed in the boiling water. The temperature of the milk increases such that $\dfrac{dy}{dt} = a\left(100 - y\right)$ where a is a positive constant. The milk's temperature when the bottle is placed into the boiling water is 5° C.

(i) Verify that $y = 100 - 95e^{-at}$ satisfies the differential equation. **1**

(ii) After two minutes, the temperature of the milk is measured to be 18° C. Find the exact value of a. **2**

(iii) Henry can be given the bottle safely when the temperature of the milk is more than 39° C. What is the minimum length of time that Melissa can leave the bottle in boiling water before it can be given to the baby safely? Answer correct to the nearest minute. **2**

(d) Use mathematical induction to prove that for integers $n \geq 1$, **3**

$$1\times 3 + 2\times 4 + 3\times 5 + ... + n\left(n+2\right) = \frac{n}{6}\left(n+1\right)\left(2n+7\right).$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the letters of the word MILLER.

(i) How many arrangements of these letters are possible if the letters are arranged in a straight line? **1**

(ii) What is the probability that the L's will be separated when the letters are arranged in a straight line? **2**

(iii) If the letters are arranged in a circle, how many arrangements are possible? **1**

(b) The probability of snow falling in the Snowy Mountains on any one of the thirty-one days in August is 0.2 **2**
Find the probability that August has exactly 10 days in which snow falls.
Give your answer as a percentage, correct to the nearest whole per cent.

(c) A particle moves along a straight line such that its distance from the origin at t seconds is x metres and its velocity is v.

(i) Prove that $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$. **2**

(ii) If the acceleration satisfies $\frac{d^2x}{dt^2} = -4\left(x + \frac{16}{x^3}\right)$, and if the particle is initially at rest when $x = 2$, show that **3**

$$v^2 = 4\left(\frac{16 - x^4}{x^2}\right).$$

(d) Assume that tides rise and fall in Simple Harmonic Motion. A ship needs 11 metres of metres of water to pass down a channel safely. At low tide, the channel is 8 metres deep and at high tide 12 metres deep. Low tide is at 10.00 am and high tide is at 4.00 pm. **4**

Find the first time after 10.00 am at which the ship can safely proceed through the channel.

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The coefficient of x^k in $(1+x)^n$, where n is a positive integer, is denoted by ${}^nC_k$. **3**

Show that

$${}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + \ldots + (n+1)\,{}^nC_n = (n+2)\,2^{n-1}.$$

(b) Let $g(x) = e^x + \dfrac{1}{e^x}$ for all real values of x and let $f(x) = e^x + \dfrac{1}{e^x}$ for $x \le 0$.

(i) Sketch the graph $y = g(x)$ and explain why $g(x)$ does not have an inverse function. **2**

(ii) On a separate diagram, sketch the graph of the inverse function $y = f^{-1}(x)$. **1**

(iii) Find an expression for $y = f^{-1}(x)$. **3**

(c) A projectile is fired from the origin O with velocity V and with angle of elevation θ, where $\theta \ne \dfrac{\pi}{2}$. You may assume that

$$x = Vt\cos\theta \quad \text{and} \quad y = -\frac{1}{2}gt^2 + Vt\sin\theta,$$

where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing.

(i) Show that the equation of flight of the projectile can be written as **2**

$$y = x\tan\theta - \frac{1}{4h}x^2\left(1+\tan^2\theta\right), \text{ where } \frac{V^2}{2g} = h.$$

(ii) Show that the point (X, Y), where $X \ne 0$, can be hit by firing at two different angles θ_1 and θ_2 provided **2**

$$X^2 < 4h(h - Y).$$

(iii) Show that no point **above** the x axis can be hit by firing at two different angles θ_1 and θ_2, satisfying $\theta_1 < \dfrac{\pi}{4}$ and $\theta_2 < \dfrac{\pi}{4}$. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \;, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

MC → Self	Q1 → AJ	Q3 → IB	
	Q2 → IM	Q4 → MV	

Y12 Mathematics Extension 1 (HSC ASSESSMENT TASK 5: TRIAL HSC) - Term 3 2013

Suggested Solution (s)	Comments

Section I (Multiple Choice)

1. $x^3 - 3^3 = (x-3)(x^2+3x+9)$ → C ✓

2. ${}^6C_3 \times {}^4C_2 = 120$ → B ✓

3. $T_{k+1} = {}^6C_k \, x^{6-k} \left(\frac{2}{x}\right)^k$

$= {}^6C_k \, 2^k \, x^{6-2k}$

$6-2k = 0, \quad k = 3$

$T_4 = {}^6C_3 \, 2^3$

$= 160$ → A ✓

4. Let $x = \frac{2y}{y+1}$

$xy + x = 2y$

$x = 2y - xy$

$x = y(2-x)$

$y = \frac{x}{2-x}$

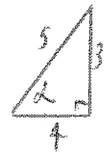
$f^{-1}(3) = \frac{3}{2-3}$ → A ✓

$= -3$

5. $\sin\theta(2\sin\theta - 1) = 0$

$\sin\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\theta = 0, \pi, \frac{\pi}{6}, \frac{5\pi}{6}$ → B ✓

6. 

$v\sin\alpha = 30 \times \frac{3}{5}$

$= 18 \text{ ms}^{-1}$ → A ✓

7. $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$

$\int \frac{1}{2} + \frac{1}{2}\cos 4x \, dx = \frac{x}{2} + \frac{\sin 4x}{8} + C$ → C ✓

8. $P(x) = (x-a)^2(b-x)(b+x)$

$= 0$ when $x = a, \pm b$

$a > b$ and $P(0) > 0$ → C ✓

9. $\sin\left(6t + \frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}$

$6t + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$

$6t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$

$t = 0, \frac{\pi}{12}, \frac{2\pi}{12}, \frac{3\pi}{12}, \ldots$

$= \frac{n\pi}{12}$, n is an integer → D ✓

10. $x + py = 2ap + ap^3$

$x = 0, y = a \rightarrow ap = 2ap + ap^3$

$0 = ap + ap^3$

$0 = ap(1+p^2)$

$p = 0$ is the only soln → B ✓

Y12 Mathematics Extension 1 (HSC ASSESSMENT TASK 5: TRIAL HSC) - Term 3 2013

Suggested Solution (s)	Comments

Section II

Q11 :

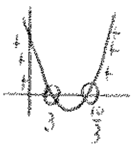
a) $\dfrac{1}{x-3} < 3$

$x-3 < 3(x-3)^2$

$3(x-3)^2 - (x-3) > 0$

$(x-3)\left[3(x-3)-1\right] > 0$

$(x-3)(3x-10) > 0$



$x < 3$ or $x > \dfrac{10}{3}$

b) $\displaystyle\int_0^3 \frac{dx}{\sqrt{3^2-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^3$

$= \sin^{-1}(1) - 0$

$= \dfrac{\pi}{2}$

c) (i) $\dfrac{dy}{dx} = \dfrac{2}{1+4x^2}$

(ii) $\dfrac{dy}{dx} = 4(\sec x)^3 \cdot \sec x \tan x$

$= 4\sec^4 x \tan x$

d) $\tan\theta = \left|\dfrac{-5}{3}\right|$

$\theta = 59^\circ$

e) $\dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \dfrac{\frac{3}{2}}{1}$

$= \dfrac{3}{2}$

f) $u = 1 + \tan x$

$du = \sec^2 x \, dx$

$x = 0, \; u = 1$

$x = \dfrac{\pi}{4}, \; u = 2$

$\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x \, dx}{\sqrt{1+\tan x}} = \int_1^2 \frac{du}{u^{\frac{1}{2}}}$

$\displaystyle = \int_1^2 u^{-\frac{1}{2}} \, du$

$= \left[2\sqrt{u}\right]_1^2$

$= 2\sqrt{2} - 2$

or

$2(\sqrt{2}-1)$

Y12 Mathematics Extension 1 (HSC ASSESSMENT TASK 5: TRIAL HSC) - Term 3 2013

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q12: a) (i) $f(x) = x - 3\sin x$ $f(2) = 2 - 3\sin 2$ $= -0.72 < 0$ $f(3) = 3 - 3\sin 3$ $= 2.5766... > 0$ ∴ Since $f(2)$ and $f(3)$ are of different sign and $f(x)$ is continuous then root lies between $x=2$ and $x=3$. (ii) $f'(x) = 1 - 3\cos x$ $x_1 = 2 - \frac{2 - 3\sin 2}{1 - 3\cos 2}$ $= 2.3237...$ $= 2.32$	✓ (Must have reason) ✓ ✓	$\angle BCE = 180° - \angle EAB$ (opposite angles of a cyclic quad. are supplementary) $= 180° - (180° - 3x°)$ $= 3x°$	✓
b) (i) $\angle AOB = \angle CBO$ (alternate angles, $AO \parallel BC$) $\angle AEB = \frac{1}{2}\angle AOB$ (angle at the centre is twice angle at circum.) $= \frac{1}{2}(2x)$ $= x$ $\angle EAB = 180° - \angle AEB - \angle ABE$ (angle sum of $\triangle AEB$) $= 180° - x° - 2x°$ $= 180 - 3x°$	✓ ✓ ✓	c) (i) $y = 100 - 95e^{-at} \rightarrow 95e^{-at} = 100 - y$ (1) $\frac{dy}{dt} = -95 \times -ae^{-at}$ $= a(95e^{-at})$ $= a(100 - y)$ from (1) (ii) $t = 2, y = 18,$ $\rightarrow 18 = 100 - 95e^{-2a}$ $-82 = -95e^{-2a}$ $e^{-2a} = \frac{82}{95}$ $-2a = \ln\left(\frac{82}{95}\right)$ $a = -\frac{1}{2}\ln\left(\frac{82}{95}\right)$ (iii) $t = ?,\ y = 39,$ $39 = 100 - 95e^{-at}$ $-61 = -95e^{-at}$ $e^{-at} = \frac{61}{95}$ $-at = \ln\left(\frac{61}{95}\right)$ $t = \frac{\ln\left(\frac{61}{95}\right)}{\frac{1}{2}\ln\left(\frac{82}{95}\right)}$ $= 6.02...$ $= 6$ minutes	✓ ✓ ✓ ✓ ✓

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q12 cont'd: d) Prove the statement is true for $n=1$ $LHS = 1(1+2)$ $= 3$ $RHS = \frac{1}{6}(1+1)(2+7)$ $= 3$ $= LHS$ $\therefore$ the statement is true for $n=1$ Assume the statement is true for $n=k$ ie $1\times 3 + 2\times 4 + \dots + k(k+2) = \frac{k}{6}(k+1)(2k+7)$ Prove the statement is true for $n=k+1$ ie $S_{k+1} = \frac{k+1}{6}(k+2)(2k+9)$ $S_{k+1} = T_{k+1} + S_k$ $= (k+1)(k+3) + \frac{k}{6}(k+1)(2k+7)$ from Assumption $= k+1\left[(k+3) + \frac{k}{6}(2k+7)\right]$ $= \frac{k+1}{6}\left[6(k+3) + k(2k+7)\right]$ $= \frac{k+1}{6}\left[2k^2+13k+18\right]$ $= \frac{k+1}{6}\left[(2k+9)(k+2)\right]$ $\therefore$ the statement is true for $n=k+1$		$\therefore$ Since the statement is true for $n=1$ and $n=k+1$, then it is true for $n=2,3,4,\dots$ for all integers $n \geq 1$.	

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q13:

a) (i) Arrangements $= \dfrac{6!}{2!}$

$= 360$

(ii) $P(\text{L's separated})$

$= 1 - P(\text{L's together})$

$= 1 - \dfrac{5!}{360}$

$= \dfrac{2}{3}$

(iii) Arrangements $= \dfrac{5!}{2!}$

$= 60$

b) $P(\text{snow}) = 0.2$, $P(\text{no snow}) = 0.8$

$P(\text{exactly 10 days of snow})$

$= {}^{31}C_{10}\,(0.8)^{21}(0.2)^{10}$

$= 0.0418894\ldots$

$= 4.18894\ldots\%$

$= 4\%$

c) (i) $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{d}{dv}\left(\dfrac{1}{2}v^2\right) \times \dfrac{dv}{dx}$

$= v \times \dfrac{dv}{dx}$

$= \dfrac{dx}{dt} \times \dfrac{dv}{dx}$

$= \dfrac{dv}{dt}$

$= \dfrac{d^2x}{dt^2}$

(ii) $\dfrac{d^2x}{dt^2} = -4\left(x + \dfrac{16}{x^3}\right)$

$\dfrac{1}{2}v^2 = \displaystyle\int \left(-4x - 64x^{-3}\right) dx$

$= -2x^2 + \dfrac{32}{x^2} + C$

$v = 0,\ x = 2 \rightarrow 0 = -8 + \dfrac{32}{4}$

$\therefore C = 0$

$v^2 = -4x^2 + \dfrac{64}{x^2}$

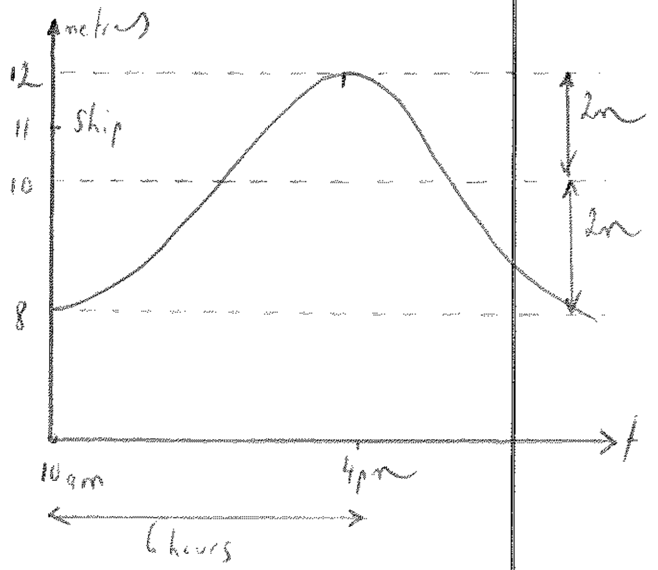
$= \dfrac{-4x^4 + 64}{x^2}$

$= \dfrac{64 - 4x^4}{x^2}$

$= 4\left(\dfrac{16 - x^4}{x^2}\right)$

Y12 Mathematics Extension 1 (HSC ASSESSMENT TASK 5: TRIAL HSC) - Term 3 2013

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments
Q13 (d)			



$\therefore$ Wavelength = 12 hours

Period = $\frac{2\pi}{n}$

$12 = \frac{2\pi}{n}$

$n = \frac{\pi}{6}$

and amplitude = 2 metres ✓

CONT'D

Question 13 (d) cont'd

Let $x(t)$ be the depth of water in the channel measured from 10:00 am where $t=0$. Then, as the motion is SIMPLE HARMONIC, $x(t)$ can be modelled by the formula :-

$$x(t) = 2\cos\left(\frac{\pi}{6}t+\alpha\right)+10$$ where 10 is the mean tidal mark. ✓

Now $x(0) = 8 \Rightarrow \cos(\alpha) = -1$ or $\alpha = \pi$ (say).

So $$x(t) = 2\cos\left(\frac{\pi}{6}t+\pi\right)+10$$
$$= 10 - 2\cos\left(\frac{\pi}{6}t\right) \; ; \left[\cos(\pi+\theta) = -\cos\theta\right]$$

We want the first occurrence for when $x(t) = 11$ as the water depth is increasing from its low tidal position.

$$\therefore \; 11 = 10 - 2\cos\left(\frac{\pi}{6}t\right) \Rightarrow \cos\left(\frac{\pi}{6}t\right) = -\tfrac{1}{2}$$ ✓
$$\Rightarrow \frac{\pi}{6}t = 2\pi/3 \text{ or } t = 4.$$

Consequently, the first opportunity to negotiate the channel safely after 10:00 am is 2:00 pm (4 hours later). ✓

OR

ALTERNATIVE APPROACH

Let $x(t)$ represent the displacement of the water level relative to its mean tidal position. Then $x(t) = 2\cos\left(\frac{\pi}{6}t+\alpha\right)$ and $x(0) = -2$. This implies $\alpha = \pi$ as before & we have $x(t) = -2\cos\left(\frac{\pi}{6}t\right)$. We want first occurrence for when $-2\cos\left(\frac{\pi}{6}t\right) = 1$ ($+1 \Leftrightarrow 11$m).

$$\Rightarrow \cos\left(\frac{\pi}{6}t\right) = -\tfrac{1}{2}$$ as before, etc.

OR

If a student models the distance function by $x(t) \;\; 2\sin\left(\frac{\pi}{6}t+\beta\right)$ then, as before, $x(0) = 8 \Rightarrow \sin(\alpha) = -1 \Rightarrow \alpha = \frac{3\pi}{2}$ (say).

So $$x(t) = 2\sin\left(\frac{\pi}{6}t+\frac{3\pi}{2}\right)+10$$
$$= 2\sin\left(\left(\frac{\pi}{6}t+\frac{\pi}{2}\right)+\pi\right)+10$$
$$= -2\sin\left(\frac{\pi}{6}t+\frac{\pi}{2}\right)+10$$
$$= -2\cos\left(\frac{\pi}{6}t\right)+10$$ as before etc!

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q14:

(a) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots + {}^nC_nx^n$ ①

$\frac{d}{dx}(1+x)^n = \frac{d}{dx}\left({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots + {}^nC_nx^n\right)$ ✓

$n(1+x)^{n-1} = {}^nC_1 + 2\,{}^nC_2x + 3\,{}^nC_3x^2 + \dots + n\,{}^nC_nx^{n-1}$ ②

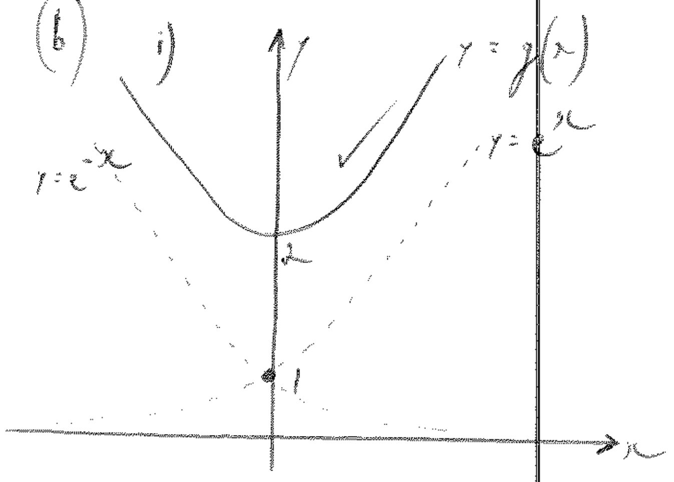
Sub. $x=1$ in ① $\Rightarrow$ $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n$ ③ ✓

Sub. $x=1$ in ② $\Rightarrow$ $n2^{n-1} = {}^nC_1 + 2\,{}^nC_2 + 3\,{}^nC_3 + \dots + n\,{}^nC_n$ ④

③ + ④ $\Rightarrow$ $n2^{n-1} + 2^n = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + 4\,{}^nC_3 + \dots + (n+1)\,{}^nC_n$ ✓

$\therefore 2^{n-1}(n+2) = {}^nC_0 + 2\,{}^nC_1 + 3\,{}^nC_2 + 4\,{}^nC_3 + \dots + (n+1)\,{}^nC_n$

(b) i)



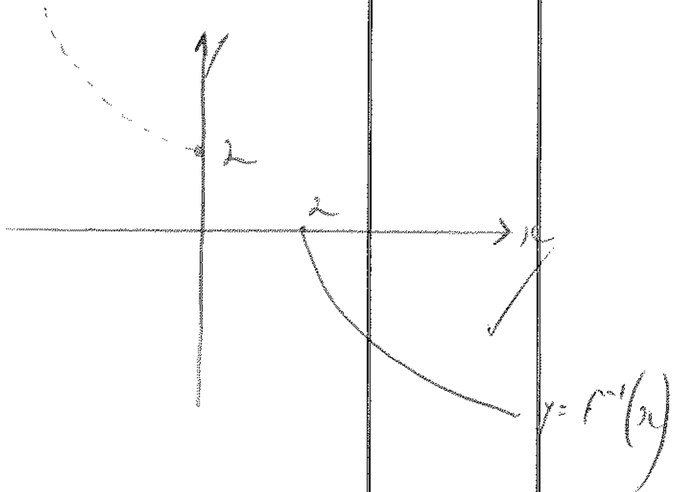
$g(x)$ does not have an inverse function because for every value of $y > 2$, there are 2 values of x. ✓

or equivalent
(– not monotonic
– not one-to-one)

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q14 cont'd:

(b) ii)



iii) Let $x = e^y + \frac{1}{e^y}$ for $y \leq 0$ and $x \geq 2$

$xe^y = e^{2y} + 1$

$0 = e^{2y} - xe^y + 1 = 0$

$0 = (e^y)^2 - x(e^y) + 1 = 0$ } $a = 1,\ b = -x,\ c = 1$ ✓

$e^y = \dfrac{-(-x) \pm \sqrt{(-x)^2 - 4(1)(1)}}{2}$

$= \dfrac{x \pm \sqrt{x^2 - 4}}{2}$ ✓

But $y \leq 0$, $\therefore e^y \leq 1$

$\therefore e^y = \dfrac{x - \sqrt{x^2 - 4}}{2}$ $(x \geq 2)$

$y = \ln\left(\dfrac{x - \sqrt{x^2-4}}{2}\right)$ $(x \geq 2)$ ✓

$= f^{-1}(x)$

Y12 Mathematics Extension 1 (HSC ASSESSMENT TASK 5: TRIAL HSC) - Term 3 2013

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q14 cont'd:

(c) i) $x = Vt\cos\theta$ (1) $\rightarrow t = \dfrac{x}{V\cos\theta}$

$y = -\frac{1}{2}gt^2 + Vt\sin\theta$ (2)

Sub. (1) in (2) $\rightarrow$ $y = -\frac{1}{2}g\left(\dfrac{x}{V\cos\theta}\right)^2 + V\left(\dfrac{x}{V\cos\theta}\right)\sin\theta$

$= \dfrac{-gx^2}{2V^2\cos^2\theta} + x\tan\theta$ ✓

$= \dfrac{-gx^2\sec^2\theta}{2V^2} + x\tan\theta$

$= \dfrac{-2gx^2\sec^2\theta}{4V^2} + x\tan\theta$

$= -\dfrac{x^2\sec^2\theta}{4h} + x\tan\theta$ ✓ $\rightarrow h = \dfrac{V^2}{2g}$

$= x\tan\theta - \dfrac{1}{4h}x^2(1+\tan^2\theta)$ $\rightarrow \sec^2\theta = 1+\tan^2\theta$.

ii) $y = x\tan\theta - \dfrac{1}{4h}x^2(1+\tan^2\theta)$

$4hy = 4hx\tan\theta - x^2(1+\tan^2\theta)$

$x^2\tan^2\theta - 4hx\tan\theta + x^2 + 4hy = 0$ (1)

Sub (X, Y) in (1) $\rightarrow$ $X^2\tan^2\theta - 4hX\tan\theta + X^2 + 4hY = 0$

Two distinct roots $\rightarrow$ $\Delta > 0$ (in $\tan\theta$)

ie $(-4hX)^2 - 4(X^2)(X^2+4hY) > 0$ ✓

$16h^2X^2 - 4X^4 - 16hX^2Y > 0$

$4h^2 - X^2 - 4hY > 0$ (since $X^2 > 0$)

cont'd

Suggested Solution (s)	Comments	Suggested Solution (s)	Comments

Q14 cont'd:

(c) ii) $4h^2 - 4hY > X^2$

$4h(h-Y) > X^2$

$X^2 < 4h(h-Y)$

$\therefore$ If $X^2 < 4h(h-Y)$ there are 2 solutions, $\tan\theta_1$ and $\tan\theta_2$, for the equation. ie 2 different angles θ_1 and θ_2 can be used to hit (X, Y).

iii) $X^2\tan^2\theta - 4hX\tan\theta + (X^2 + 4hY) = 0$ from ii)

where $\tan\theta_1$ and $\tan\theta_2$ are solutions.

$\therefore \tan\theta_1 \tan\theta_2 = \dfrac{X^2 + 4hY}{X^2}$ using product of roots

$\therefore \tan\theta_1 \tan\theta_2 = 1 + \dfrac{4hY}{X^2}$.

$\therefore \tan\theta_1 \tan\theta_2 > 1$, since $Y > 0$

But if both $0 < \theta_1 < \frac{\pi}{4}$ and $0 < \theta_2 < \frac{\pi}{4}$, then $0 < \tan\theta_1 < 1$ and $0 < \tan\theta_2 < 1$ and so $\tan\theta_1 \tan\theta_2 < 1$.

$\therefore$ No point with $Y > 0$ can be hit from 2 different angles θ_1 and θ_2 satisfying $\theta_1 < \frac{\pi}{4}$ and $\theta_2 < \frac{\pi}{4}$.

Question 14

Part (c) (iii)

Alternatively, if we suppose $\tan\theta_1$ & $\tan\theta_2$ are distinct solutions with $0 < \theta_1, \theta_2 < \frac{\pi}{4}$, then $0 < \tan\theta_1, \tan\theta_2 < 1$ & $\sum\alpha < 2$. However, from $X^2 \cdot \tan^2\theta - 4hX \cdot \tan\theta + (X^2 + 4hY) = 0$, we have $\sum\alpha = \frac{4h}{X}$. So $\frac{4h}{X} < 2 \Rightarrow X > 2h$ or $X^2 > 4h^2$ (*)
From part (ii), $X^2 < 4h(h-Y)$, i.e., $X^2 < 4h^2 - 4hY$. Now $4hY > 0$ since $h = \frac{v^2}{2g} > 0$ & the point (X, Y) is above the x-axis from data. So, $4h^2 - 4hY < 4h^2$ & consequently $X^2 < 4h^2$ (**) ($X^2 < 4h^2 - 4hY < 4h^2 \Rightarrow X^2 < 4h^2$ ~ Trichotomy property for inequalities). From (*) & (**) we have
$X^2 > 4h^2$ & simultaneously $X^2 < 4h^2$. This is impossible!
We have a contradiction and accordingly the initial supposition is false.